Vibrations of Laminated Beams

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Theme

DERIVED are equations governing the vibrations of laminated beams. The method presented is less restrictive than ones known in the literature since neither the normal stresses in the core, nor the shear stresses in faces are neglected.

Contents

This paper is a continuation of Ref. 1 and its extension to dynamic problems. As in Ref. 1 the underlying assumptions about the kinematics of the beam deformation are: a) deformations are elastic and small such that the linear elasticity theory is applicable; b) perfect bond is realized between adjacent laminae; c) Bernoulli's hypothesis of planar cross sections is valid for each lamina independently such that after the deformation each cross section consists of a number of planes interconnected at interface lines; d) deformation is symmetric with respect to vertical plane (no torsion).

As shown in Ref. 1 on the basis of these assumptions it is possible to develop a simple and consistent theory without the introduction of additional simplifications (such as neglecting the bending rigidity of the core, etc.).

Consider a laminated beam with x axis being the loci of all cross-sectional centroids and denote axial displacement by $u(x, z)$ and transverse displacement by $w(x, z)$. As a consequence of assumptions about the deformation kinematics, set forth above, both displacement components $u(x, z)$ and $w(x, z)$ are defined as a polygonal graph with vertices at extreme fibers of all three laminae. It is, for example, obvious that to determine the displacement component $u(x, z)$ we ought to know the values of the axial displacement at four vertices at each cross section. Hence, we write

$$u(x, z) = \sum_{i=1}^{4} U_i(x)\Phi_i(z) \qquad (1)$$

where $U_i(x)$ are unknown generalized coordinates and $\varphi_i(z)$ are corresponding predetermined (chosen by us) deformation modes. For the other displacement component $w(x, z)$ in the same way it follows

$$w(x, z) = \sum_{j=1}^{2} W_j(x)\Psi_j(z) \qquad (2)$$

where $W_j(x)$ are generalized coordinates and $\Psi_j(z)$ deformation modes.

Instead of choosing an arbitrary set of six independent modes it is obviously advantageous if they are chosen to be mutually orthogonal functions. As shown in Ref. 1 they may be chosen such that the modes φ_1, φ_2 and ψ_1 correspond to the conventional beam analysis while the other three reflect the distortion of the cross section.

For the chosen set of function φ_1 and ψ_k, from the principle of minimum potential energy it is possible to write two independent sets of equations with constant coefficients governing the laminated beam vibrations. Firstly, obtained is a set of three ordinary differential equations governing the motion for which the axis of the beam remains straight

$$D_x u'' + v_c D_{cx}\chi' - \rho \bar{A} \ddot{u} - \rho A_c(1 - \mu_1)\ddot{\omega} = 0 \qquad (3a)$$

$$-v_c D_{cx} u' - (v_c D_{cx} + G_f K_3)\omega' + G_f K_s \chi'' - D_{cx}\chi - \rho \bar{K}_s \ddot{\chi} = 0 \qquad (3b)$$

$$D_\omega \omega'' + (v_c D_{cx} + G_f K_3)\chi' - G_f K_4 \omega - \rho \bar{A}_\omega \ddot{\omega} - \rho A_c(1 - \mu_1)\ddot{u} = 0 \qquad (3c)$$

where $u(x)$ is the axial displacement, $\omega(x)$ axial distortion and $\chi(x)$ the contraction of the core.

Secondly, obtained is a set of ordinary differential equations with constant coefficients governing the predominantly flexural (with shear) behavior of the beam.

$$D_z \varphi'' - G_f A_e[\varphi + w' + (K_1/A_e)\psi] - \rho I \ddot{\varphi} + \rho I_{\varphi\psi} \ddot{\psi} = 0 \qquad (4a)$$

$$G_f A_e[\varphi' + w'' + (K_1/A_e)\psi'] - \rho A \ddot{w} = 0 \qquad (4b)$$

$$D_\psi \psi'' - G_f K_1[\varphi + w' + (K_2/K_1)\psi] - \rho \bar{I}_\psi \ddot{\psi} + \rho I_{\varphi\psi} \ddot{\varphi} = 0 \qquad (4c)$$

where $w(x)$ is the vertical displacement, φ rotation of the cross section and ψ the distortion. First two equations with $\psi \equiv 0$ are apparently well known equations of the Timoshenko beam. Cross-sectional parameters in Eqs. (2) and (3) are defined in Ref. 1 except for

$$\bar{K}_s = h_c^2(\tfrac{1}{3}A_c + A_f)$$

$$I_{\varphi\psi} = \tfrac{1}{6}h_c h_f\{2h_c A_c + A_f[h_c(2 - n) + h(1 - 2n)]\}$$

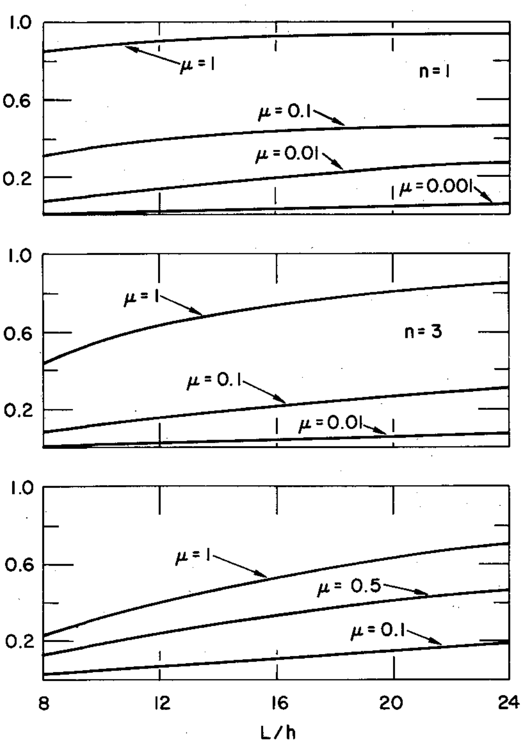

Fig. 1 Nondimensional flexural frequencies.

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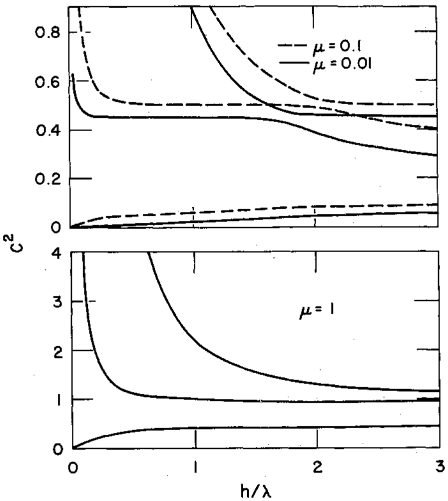

Fig. 2 Phase-velocity diagram for flexural vibrations.

Boundary conditions are also directly deducible from the employed variational principle. For instance for a cross section simply supported (in the sense of bending) and free to distort it follows

$$w = w'' = \varphi' = \psi' = 0$$

From Eqs. (4) we in a routine way derive the frequency determinant as follows

$$\begin{vmatrix} -\alpha_n^2 \dfrac{E_f I_e}{G_f K_1} - \dfrac{A_e}{K_1} + \dfrac{E_f I^2 \alpha_n^4}{G_f K_1 A}\Omega^2 & -\alpha_n \dfrac{A_e}{K_1} & -1 - \dfrac{E_f I I_{\varphi\psi}\alpha_n^4}{G_f K_1 A}\Omega^2 \\ -\alpha_n \dfrac{A_e}{K_1} & -\dfrac{A_e}{K_1}\alpha_n^2 + \dfrac{E_f I \alpha_n^4}{G_f K_1}\Omega^2 & -\alpha_n \\ -1 - \dfrac{E_f I I_{\varphi\psi}\alpha_n^4}{G_f K_1 A}\Omega^2 & -\alpha_n & -\dfrac{E_f I_\psi \alpha_n^2}{G_f K_1} - \dfrac{K_2}{K_1} + \dfrac{E_f I \bar{I}_\psi}{G_f K_1 A}\alpha_n^4 \Omega^2 \end{vmatrix} = 0 \tag{5}$$

where the natural frequency in nondimensionalized with the natural frequency of the conventional beam theory $(\alpha_n^4 E_f I/\rho A)^{1/2}$.

Nondimensional frequencies Ω^2 are computed from the frequency determinant (5), for $0 \leqq \mu \leqq 1.0$ (with $v_c = v_f = 0.2$; thus $\mu_1 = \mu_2 = \mu$) and for a range of ratios $L/h = 8, 12, 16, 20$. The results for first, third and fifth vibration modes are plotted in Fig. 1. As expected, the higher the value of μ is, the more the laminated beam approaches the conventional solid beam with distortionless cross section ($\psi \equiv 0$). The difference between laminated and solid beam is more pronounced for higher modes and shorter beams, i.e., where the influence of shear stresses in comparison with bending stresses if more felt. The phase-velocity diagram is presented in Fig. 2.

For the case of the motion for which the beam axis remains straight, the frequency determinant is derived to be

$$\begin{vmatrix} -D_x\alpha_n^2 + \rho A \Omega_1^2 & \rho A_c(1-\mu_1)\Omega_1^2 & \alpha_n v_c D_{cx} \\ A_c(1-\mu_1)\Omega_1^2 & -D_\omega \alpha_n^2 - G_f K_4 + \rho \bar{A}_\omega \Omega_1^2 & (v_c D_{cx} + G_f K_3)\alpha_n \\ \alpha_n v_c D_{cx} & (v_c D_{cx} + G_f K_3)\alpha_n & -\alpha_n^2 G_f K_s - D_{cx} + \rho \bar{K}_s \Omega_1^2 \end{vmatrix} = 0 \tag{6}$$

It turns out that the influence of cross-sectional distortion of the conventional natural axial vibration frequency is very small. The bulging frequency is rather high even for very soft cores, while the contraction (breathing) frequency is much smaller especially for low values of μ. Their dependence on the parameter L/h is more-or-less identical.

In order to obtain an approximate formula for the flexural frequency we neglect rotatory and shear distortion inertia terms ($\rho I \ddot{\varphi} = \rho \bar{I}_\psi \ddot{\psi} = 0$) in Eq. (4). In this case the following equation is derived

$$w^{vi} - k^2 w^{iv} - \frac{\rho L^2 A}{G_f A_e}\ddot{w}^{iv} + \frac{\rho L^4 A}{D_\psi A_e}\left(K_2 + \frac{D_\psi}{D_z}A_e\right)\ddot{w}'' - k^2 \frac{\rho A L^4}{D_z}\ddot{w} = 0 \tag{7}$$

where the differentiation is with respect to the nondimensional coordinate $\xi = x/L$, with L being a characteristic length (say, the span). k^2 is the shear coefficient (see Ref. 1).

For an infinite beam (or a simply supported one)

$$[\alpha_1^4 E_f I/G_f A_e L^2 + \alpha_1^2(IK_2/I_\psi A_e + I/I_e) + k^2 I/I_e]\Omega^2 = k^2 + \alpha_1^2$$

where Ω is again nondimensionalized with the natural frequency of a conventional beam.

In conclusion, it appears that the presented theory is, in comparison with existing ones: less restrictive and, therefore, applicable to a wider class of problems (core ought not be soft, faces thin, etc.); incorporating both axial and transverse motions; an extension of the Timoshenko beam theory.

Reference

[1] Krajcinovic, D., "Sandwich Beam Analysis," *Journal of Applied Mechanics*, to be published.